The Riemann Zeta Club

Topic 3: Algebra

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Topic 3 – Algebra

Part 1: Recursive Expressions and Expansion Identities

Recursive Expressions

Sometimes values are defined in terms of themselves.

What is the value of the following?

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

?

Recursive Expressions

Question:

$$A = \sqrt{2007 + \sqrt{2007 + \sqrt{2007 + \cdots}}}$$

$$B = \sqrt{2007 - \sqrt{2007 - \sqrt{2007 - \cdots}}}$$

Without explicitly calculating A or B, find $A - B$.

?

Using helpful expansion identities

In solving algebraic problems, it's often immensely useful to find expansions that involve the terms we know and that we're trying to find. For example:

$$(A+B)^2 = A^2 + 2AB + B^2$$

would clearly be useful if we knew $A+B$ and A^2+B^2, and wanted to find AB.

$$(A+1)(B+1) = AB + A + B + 1$$

We used this earlier in the Number Theory module when our equation involved A, B and AB and we wanted to factorise.

Using helpful expansion identities

Question: We earlier found equations $A^2 = 2007 + A$ and $B^2 = 2007 - B$ and that $A - B = 1$. **Now find $\boldsymbol{AB}$.**

$$A = \sqrt{2007 + \sqrt{2007 + \sqrt{2007 + \cdots}}}$$

$$B = \sqrt{2007 - \sqrt{2007 - \sqrt{2007 - \cdots}}}$$

?

Using helpful expansion identities

[SMC] Four positive integers a, b, c and d are such that $abcd + abc + bcd + cda + dab + ab + bc + cd + da + ac + bd + a + b + c + d = 2009$. What is the value of $a + b + c + d$?

A: 73

B: 75

C: 77

D: 79

E: 81

The expansion we want to use is
$(a+1)(b+1)(c+1)(d+1) - 1 = LHS$
So $(a+1)(b+1)(c+1)(d+1) = 2010 = 2 \times 3 \times 5 \times 67$
So a, b, c and d must have the values 1, 2, 4 and 66 (in any order). The sum is 73.

Factorising Strategies

Factorise $a^2 + b^2 + c^2 - ab - bc - ca$

We might think to get these terms, we'd have something like $(a-b)^2 = a^2 + b^2 - 2ab$. Either we'd need to add an ab, or we'd have to add an a^2 and b^2 before halving it.

The expression is clearly symmetrical (in the sense we can interchange a, b, c and have the same expression), so let's try

$$(a-b)^2 + (b-c)^2 + (c-a)^2$$

This gives $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

Thus our factorisation is $\frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2$

Factorising Strategies

Factorise $a^3 + a^2b + ab^2 + b^3 + ac^2 + bc^2$

Notice that there's a symmetry here between a and b, but not with c. We'll have this same symmetry/asymmetry in the factorised expression.

We might attempt to start factorising like so, which would give us the first four terms in the expansion…

$$(a^2 + b^2 + c^2)(a \quad + \quad b \qquad)$$

…and we can see that adding this extra term in the first bracket would give us the remaining terms.

Factorising more difficult expressions is ultimately mostly about intelligent guess work, just by considering how terms combine across brackets.

Factorising Strategies

Try factorising these:

$$x^3 + x^2 + xy^2 + y^2 = ?$$

$$a^3 - a^2 + a - 1 = ?$$

$$ab + bc - 3a - 3c = ?$$

Pro Tip: You can check your factorisations by trying small values for your variables, e.g. $x = 0$ or 1 or -1. This doesn't guarantee it's correct (you might have got lucky with the values you chose!) but at least gives you greater confidence in its validity.

Using helpful expansion identities

[BMO1] Find all integer solutions x, y and z for:
$x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$

You might be tempted to try and use $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$, but this leaves the terms xy and zx which we're unable to deal with.

So looking at the first equation, what terms could we combine into some factorised expression?

?

Calculating big numbers

Often algebra can be used to determine large values without a calculator. Replace numbers with variables and manipulate.

A quickie: What is 99^2?

?

A quickie: What is 101^3?

?

What about 101^4?

?

Calculating big numbers

[BMO1] Find the value of

$$\frac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}$$

Solution:

?

Making an Effective Substitution

In many cases (particularly Olympiad problems), it's worthwhile making a substitution that simplifies our problem.

Question: Find real solutions to $(x+y)^2 = (x+3)(y-3)$

What substitution might aid us? (look at the relationship between the contents of the three brackets)

?

?

Topic 3 – Algebra

Part 2: Simultaneous Equations & Surds

Surd Manipulation

Question: Find the least positive integer n such that:

$$\sqrt{n+1} - \sqrt{n} < 10^{-3}$$

?

Surd Manipulation

[SMC] Which of the following is equal to:

$$\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}}$$

A: $\sqrt{1003}-\sqrt{1002}$

B: $\sqrt{1005}-\sqrt{1004}$

C: $\sqrt{1007}-\sqrt{1005}$

D: $\sqrt{2005}-\sqrt{2003}$

E: $\sqrt{2007}-\sqrt{2005}$

Hint: Perhaps we can reorganise the contents of the outer square root such that it's a squared expression?

Surd Manipulation

[SMC] Which of the following is equal to:

$$\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}}$$

$$=\frac{1}{\sqrt{2005+\sqrt{(2005+1)(2005-1)}}}=\frac{1}{\sqrt{2005+2\sqrt{1003}\sqrt{1002}}}$$

At this point, since we have twice the product of two things, it suggests we can perhaps get the denominator in the form $(a+b)^2=a^2+2ab+b^2$, so that we can take the square root. Indeed this works very nicely:

$$=\frac{1}{\sqrt{1003+2\sqrt{1003}\sqrt{1002}+1002}}=\frac{1}{\sqrt{\left(\sqrt{1003}\right)^2+2\sqrt{1003}\sqrt{1002}+\left(\sqrt{1002}\right)^2}}$$

$$=\frac{1}{\sqrt{\left(\sqrt{1003}+\sqrt{1002}\right)^2}}=\frac{1}{\sqrt{1003}+\sqrt{1002}}$$

Now we can just rationalise the denominator to get $\sqrt{1003}-\sqrt{1002}$.

Solving with Surds

For what x do the following equality hold:

$$\sqrt{x^2} = x$$

?

Solving with Surds

What about:

$$\sqrt{(1-x)^2} = 1 - x$$

?

Solving with Surds

[Source: UKMT Mentoring]

Find the values of x for which:

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x+\sqrt{2x-1}}=\sqrt{2}$$

?

Simultaneous Equations

What are the two main ways you would solve simultaneous equation?

$$x + 2y = 3 \quad (1)$$
$$x - 2y = 7 \quad (2)$$

?

$$x + 2y = 3 \quad (1)$$
$$x^2 + y^2 = 2 \quad (2)$$

?

Simultaneous Equations

BMO questions essentially only use these two principles. You just have to be a bit more creative.

[BMO1] Solve the simultaneous equations where x, y, z are **integers**:

$$x + y - z = 12 \qquad (1)$$

$$x^2 + y^2 - z^2 = 12 \qquad (2)$$

Hint: What should we substitute, and why?

?

Simultaneous Equations

Sometimes we add/subtract equations to be able to get the RHS as 0 and be able to factorise the LHS.

[BMO1] Solve for real x, y, z:

$$(x+1)yz = 12 \quad (1)$$
$$(y+1)zx = 4 \quad (2)$$
$$(z+1)xy = 4 \quad (3)$$

Hint: They made two of the numbers a 4 for a reason!

?

Simultaneous Equations

Sometimes we can **exploit symmetry**.

[BMO1] Solve for real x, y, z:

$$ab + c + d = 3 \quad (1)$$
$$bc + d + a = 5 \quad (2)$$
$$cd + a + b = 2 \quad (3)$$
$$da + b + c = 6 \quad (4)$$

Hint: Notice that $3 - 2 = 6 - 5$

?

?

Summary of Tips

1. You can add and subtract equations:
    a. When you require **integer solutions**, the resulting expression after adding/subtracting might be factorisable in the form $(..)(..) = c$, where you can then reason about possible factor pairs of c.
    b. When your require **real solutions**, factorising in the form $AB = 0$ helps, because then we can say $A = 0$ or $B = 0$.
    c. The above might yield the relationship between just two variables. In which case, you could always then substitute into the original equations to eliminate one variable.
2. You can use elimination when you can one variable in terms of others.
3. **Spot patterns** in constants on the RHS. This might give clues as to what equations would be good to add/subtract. Add/subtract based on what variables you want in the resulting equation, rather than just arbitrarily.

Topic 3 – Algebra

Part 3: Inequalities

Starter

Solve $2x \geq 4$ → ?

Solve $-2x \geq 4$ → ?

Dividing or multiplying by a negative number flips the direction of the inequality.

Solve $\frac{1}{x} \geq 2$ → ?

You might think you can do $1 \geq 2x$ and hence $x \leq \frac{1}{2}$. You're not allowed to multiply both sides by x though, because you don't know whether it's positive or negative, and hence it may or may not flip the direction! You'll learn how to solve these in FP2, but in summary, we could just consider where for the graph $y = \frac{1}{x}$, the curve has a y value at least 2.

Forming inequalities using areas and lengths

By bounding a curve between two straight lines, or vice versa, we can often form some interesting inequalities by comparing lengths and areas.

Question: a) Using suitable squares and circles, show that $2\sqrt{2} < \pi \leq 4$.

?

Forming inequalities using areas and lengths

By bounding a curve between two straight lines, or vice versa, we can often form some interesting inequalities by comparing lengths and areas.

Question: b) Show also (by perhaps using other shapes) that $3 < \pi \leq 2\sqrt{3}$.

?

Solving Inequalities using Positive Terms

Prove that $x^2 + y^2 \geq 2xy$ for all real x, y.

?

Using this trick in Number Theory problems

[BMO1] Find all integer solutions x, y and z for:
$x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$

Earlier we found we could simplify the first equation to: $x^2 + (y - z)^2 = 2$

What can we determine about x and $y - z$?

?

Solving Inequalities using Positive Terms

Practice Questions:

Prove that $a^2 + 4b^2 \geq 4ab$ → ?

Prove that $x^2 + \frac{1}{x^2} \geq 2$ → ?

(Harder!) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$

→ ?

Using a substitution

Sometimes making a substitution makes solving an inequality easier to solve:

Prove that $a + b + 1 \geq 2\sqrt{a + b}$

Make the substitution: $x =$?

Then:

?

Types of Mean

$A \quad B \quad C$

Arithmetic Mean

?

Harmonic Mean

?

Geometric Mean

?

AM-GM Inequality

This is a hugely helpful inequality comparing Arithmetic and Geometric Means:

Arithmetic Mean $\geq$ **Geometric Mean**

$$\frac{x+y}{2} \geq \boxed{?}$$

So we can form an inequality using any sum:

$$x + y \geq \boxed{?}$$

How could we easily prove the above using a technique we've seen before?

AM-GM Inequality

Examples:

$$x^2 + y^2 \geq \boxed{?}$$

$$x + y + z + w \geq \boxed{?}$$

$$x^2y + y^2z + z^2x \geq \boxed{?}$$

$$x + 1 \geq \boxed{?}$$

$$(x+1)(y+1)(z+1) \geq \boxed{?}$$

AM-GM Inequality

This helps us prove certain inequalities.

Prove that $(a+b)(b+c)(c+a) \geq 8abc$.

?

Prove that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$

?

Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is really quite awesome.
Its form is as such:

$$\left(\mathbf{x}_1^2+\mathbf{x}_2^2+\cdots+\mathbf{x}_n^2\right)\left(\mathbf{y}_1^2+\mathbf{y}_2^2+\cdots+\mathbf{y}_n^2\right) \geq \left(\mathbf{x}_1\mathbf{y}_1+\cdots+\mathbf{x}_n\mathbf{y}_n\right)^2$$

This looks rather horrible, but constructing such an inequality is simple if done is a certain way…

Cauchy-Schwarz Inequality

Example:

STEP 1: Start with a pair of brackets on the LHS, and a single squared bracket on the RHS.

$$(x^4+x^2+1)(1+y^2+1) \geq (x^2+xy+1)^2$$

STEP 2: Put some sum in the RHS.

STEP 3: For each term in the sum, square it, then think of two terms which multiply to give this.

Cauchy-Schwarz Inequality

We can be slightly creative with our sum.

"Prove that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9$"

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq (1+1+1)^2$$

By the way, this gives us a generally handy inequality of:

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{x+y+z}$ and similarly $\frac{1}{x}+\frac{1}{y} \geq \frac{4}{x+y}$.

In general, $\sum \frac{\mathbf{1}}{\boldsymbol{x_i}} \geq \frac{\boldsymbol{n^2}}{\sum \boldsymbol{x_i}}$

Cauchy-Schwarz Inequality

Given that a, b, c are positive real numbers such that $a + b + c = x + y + z$, prove that $\frac{a^2}{y+z} + \frac{b^2}{x+z} + \frac{c^2}{x+y} \geq \frac{a+b+c}{2}$

?

Cauchy-Schwarz Inequality

Let x, y, z be real numbers such that $xyz = 1$.
Prove that $(x^2+1)(y^2+1)(z^2+1) \geq \left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right)$

(Hint: What is $1+\frac{x}{y}$ as a single fraction?)

?

?

Inequality Proofs using Geometry

Sometimes there's non-traditional and seemingly barmy ways of proving inequalities. The following problem is on your worksheet:

[BMO1] Prove that for all real x, y, z, that

$$(x^2 + y^2)^2 \geq (x + y + z)(x + y - z)(x + y - z)(x + y - z)$$

On the worksheet I recommend using the Trivial Inequality. But the BMO model solutions mention that one student's solution used **Heron's Formula** for the area of a triangle.

Let's try it!

Recall that $T = \sqrt{s(s-a)(s-b)(s-c)}$

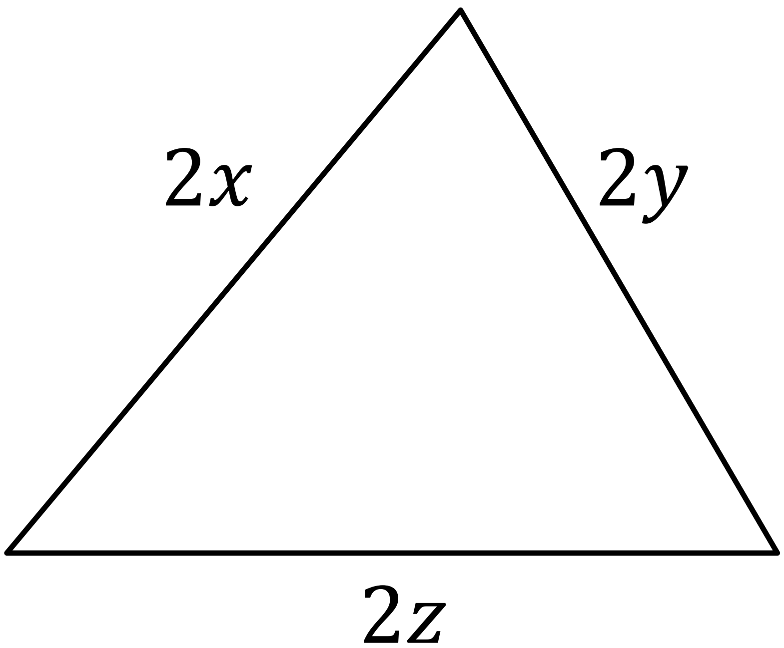
Where $s = \frac{1}{2}(a + b + c)$

Coming Soon! This section on inequalities in geometric problems will soon by hugely expanded to help tackle the very hardest BMO1 problems.

Inequality Proofs using Geometry

Prove that for all real x, y, z, that

$$(x^2 + y^2)^2 \geq (x + y + z)(x + y - z)(x + y - z)(x + y - z)$$



Notice that from $(x + y + z)$ to $(x + y - z)$, the decrease is $2x$. Since in the formula we have s and $s - c$. This suggests we should let the sides be $2x, 2y$ and $2z$. Then in Heron's formula:

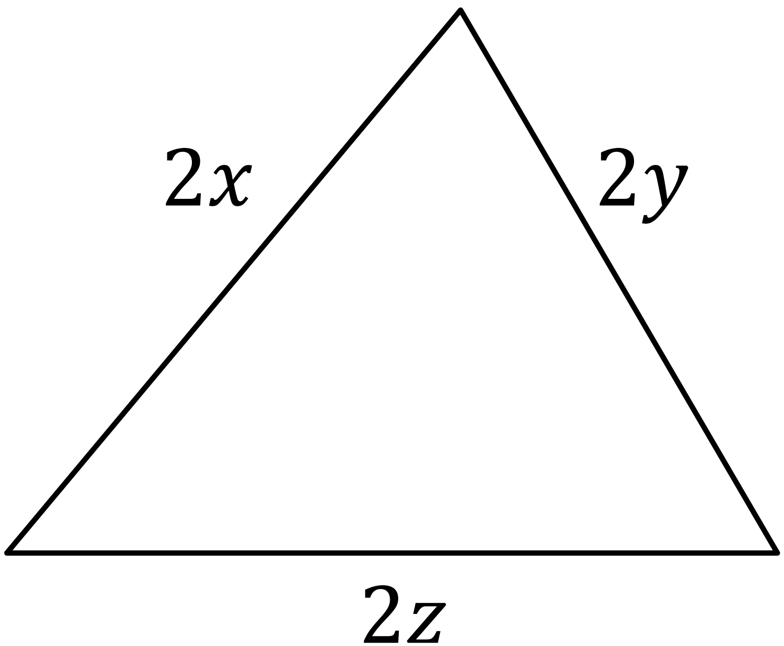
$$s = \frac{1}{2}(a + b + c) = x + y + z$$

If we square root both sides of the inequality, the RHS matches exactly. Now we have to think about what area our triangle could be less than:

Inequality Proofs using Geometry

Prove that for all real x, y, z, that

$$(x^2+y^2)^2 \geq (x+y+z)(x+y-z)(x+y-z)(x+y-z)$$



We could also use the formula:

$$\frac{1}{2}ab\sin C = \frac{1}{2}(2x)(2y)\sin C$$
$$= 2xy\sin C$$

for the area of the triangle.

We can form the inequality:

$$2xy\sin C \leq \boxed{?}$$

So far we've shown that using Geometry that:

$$2xy \geq \sqrt{(x+y+z)(x+y-z)(x+y-z)(x+y-z)}$$

All that remains is to show that $x^2+y^2 \geq 2xy$.

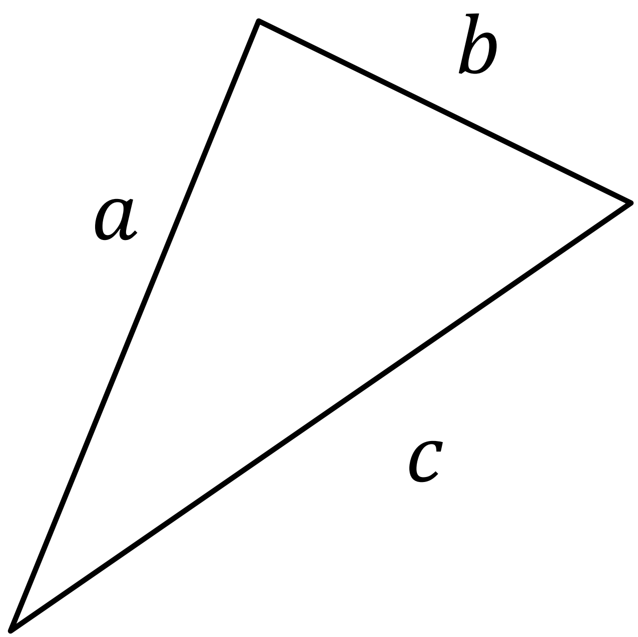
This gives $(x-y)^2 \geq 0$, which is true by the Trivial Inequality.

Inequality Proofs using Geometry

We saw that one potentially useful geometrical inequality is:

$$Area \leq \frac{1}{2}ab$$

Another useful one is the **Triangle Inequality**.



Triangle Inequality

$$a < b + c$$

i.e. Each side is less than the sum of the other two sides.
(What would happen if they were equal?)

Summary

There's 3 main approaches to solving inequalities:

1. Trivial Inequality

Put your inequality in the form $X^2 + Y^2 + \cdots \geq 0$
This is trivially true because the sum of squares is positive.
You may need a **creative factorisation**.

2. AM-GM Inequality

The arithmetic mean is greater or equal to the geometric mean.
e.g.

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad a+b+c \geq 3\sqrt[3]{abc}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}}$$

Summary

There's 3 main approaches to solving inequalities:

3. Cauchy-Schwarz

e.g. $(a^2+1)(1+b^2) \geq (a+b)^2$

$$\left(\frac{1}{x}+\frac{1}{y}\right)(x^2+y^2) \geq (\sqrt{x}+\sqrt{y})^2$$

However, there are a number of other inequality theorems, e.g. Rearrangement and Chebyshev inequalities, that we won't explore here. You can find more info here: http://www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf